Introduction to Game Theory

CMPS 17

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Game Theory

- Wikipedia: "Game theory is a study of strategic decision making"
- Father of game theory: John von Neumann (1903-1957) Hungarian mathematician.

- 1944 book *Theory of Games and Economic Behavior*

What is a game?

- A set of participants (called players)
- Each player has options (strategies) about how to behave.
- For each choice in strategies, each player receives a *payoff* that can depend on the strategies of the other players.

Simple game

Mow the lawn or get an ice cream?

You and your brother have a choice, mow the lawn or get an ice cream.

We will rate happiness on a scale of 1-10.

- If you get an ice cream, but your brother doesn't, it will be fun and you know that the lawn is being mowed. Your happiness level will be a 9.
- If you mow the lawn, but your brother gets and ice cream, your happiness will be a 4.
- If you both get an ice cream, your parents will get mad at you. Your happiness will be an 7.
- If you both mow the lawn, then you feel good about being hardworking and your happiness is an 8.

Your brother rates happiness on the exact same scale as you.

Ice cream or mow lawn?

		Your brother: Ice cream	Your brother: Mow lawn
You	Ice cream	$7, 7$	$9, 4$
	Mow lawn	$4, 9$	$8, 8$

This figure is called a *payoff matrix*. The first entry in each cell is your payoff and the second entry is the payoff of your brother (or opponent).

Basic assumptions

- The payoffs are a complete description of everything that is important to each player.
- Each player knows the structure of the game (the payoff matrix).
- Each player acts *rationally*. The players make choices to maximize their payoff.

Ice cream or mow lawn?

		Your brother: Ice cream	Your brother: Mow lawn
You	Ice cream	$7, 7$	$9, 4$
	Mow lawn	$4, 9$	$8, 8$

How should you behave?

- If your brother chooses ice cream, you should choose ice cream.
- If your brother chooses to mow the lawn, you should choose ice cream.

So, no matter what. You should choose ice cream.

Definition

A *strictly dominant strategy* is a strategy for a player that is strictly better than all other options, regardless of what the other player does.

Strictly dominant strategy

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But in this game, is not the best possible outcome for both players! If they could communicate, they could both have a payoff of 8 rather than 7.

Prisoner's Dilemma

- 2 suspects.
- If both confess - 4 years in prison.
- If neither confesses - 1 year in prison.
- If one confesses and the other doesn't, the one that confessed gets off and the other must stay in prison for 10 years.

		Suspect 2: Confess	Suspect 2: Don't confess
Suspect 1	Confess	$-4, -4$	$0, -10$
	Don't confess	$-10, 0$	$-1, -1$

Confessing is the strictly dominant strategy.

Formalize our terminology

Definition

A *best response* is the best choice for one player given a belief about what the other player will do.

- Suppose S is the strategy chosen by player 1 and T is the strategy chosen by player 2.
- Denote the payoff to player 1 by $P_1(S, T)$ and the payoff to player 2 by $P_2(S, T)$.

Definition

Strategy S for player 1 is a *best response* to strategy T for player 2 if $P_1(S, T) \geq P_1(S', T)$ for all other strategies S' for player 1.

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Strategy S for player 1 is a *best response* to strategy T for player 2 if $P_1(S, T) \geq P_1(S', T)$ for all other strategies S' for player 1.

Definition

Strategy S for player 1 is a *strict best response* to strategy T for player 2 if $P_1(S, T) > P_1(S', T)$ for all other strategies S' for player 1.

Definition

A *dominant strategy* for player 1 is a strategy that is a best response to every strategy of player 2.

Definition

A *strictly dominant strategy* for player 1 is a strategy that is a strict best response to every strategy of player 2.

Game without two strictly dominant strategies

- 2 firms plan to make a new product.
- The market is divided into 2 groups: people who buy upscale products (40% of the market) and people who buy low-priced products (60% of the market).
- The profit that either firm makes for the high priced or low-priced product is the same, so we only need to keep track of sales.
- Firm 1 is a more popular firm so if Firm 1 and Firm 2 compete, Firm 1 will get 80% of the sales while Firm 2 will get 20% of the sales.

Game without two strictly dominant strategies

- If the 2 firms target different market groups then they get all the sales for that group. The one that targets the low-priced group gets a payoff of .6 and the one that targets the upscale group gets a payoff of .4.
- If both firms target the low-priced group. Firm 1 gets 80% of .6 (.48). Firm 2 gets 20% of .6 (.12).
- If both firms target the upscale group. Firm 1 gets 80% of .4 (.32). Firm 2 gets 20% of .4 (.08).

		Firm 2	
		Low-priced	Upscale
Firm 1	Low-priced	.48, .12	.60, .40
	Upscale	.40, .60	.32, .08

Game without two strictly dominant strategies

		Firm 2	
		Low-priced	Upscale
Firm 1	Low-priced	.48, .12	.60, .40
	Upscale	.40, .60	.32, .08

- Firm 1 has a strictly dominant strategy: Low-priced.
- Firm 2 does not have a dominant strategy.
- We can still predict that Firm 2 will play upscale since it knows that Firm 1 will play Low-priced.

Game with no dominant strategy

- If both firms approach the same client, then the client will give half its business to each.
- Firm 1 is small, so it needs Firm 2's help. If Firm 1 approaches a client on its own it will get a payoff of 0.
- If Firm 2 approaches B or C on its own, it will get their full business.
- A will only do business if both firms approach A.
- A's business is worth 8 total (4 each). B and C 's business is worth 2.

Game with no dominant strategy

		Firm 2: A	B	C
Firm 1	A	$4, 4$	$0, 2$	$0, 2$
	B	$0, 0$	$1, 1$	$0, 2$
	C	$0, 0$	$0, 2$	$1, 1$

What are the best responses?

Nash Equilibrium

- John Forbes Nash Jr. born 1928

- In his doctoral thesis he defined the notion of a Nash equilibrium.
- He shared the Nobel prize in 1994 for this concept.

Nash Equilibrium

Definition

Suppose player 1 chooses strategy S and player 2 chooses strategy T. The pair of strategies (S, T) is a *Nash equilibrium* if S is the best response to T and T is the best response to S.

		Firm 2		
		A	B	C
	A	$4, 4$	$0, 2$	$0, 2$
Firm 1	B	$0, 0$	$1, 1$	$0, 2$
	C	$0, 0$	$0, 2$	$1, 1$

(A, A) is a Nash equilibrium.

Coordination games

Definition

A *coordination game* is a game in which it is in both players best interest to coordinate on the same strategy.

Scenario: You and a partner are working on a physics problem together.

- You can either use standard or metric measurements.
- If you don't use the same system you can't combine your measurements, so you can't complete the problem.
- Neither of you have a preference.

Coordination games

Definition

A *coordination game* is a game in which it is in both players best interest to coordinate on the same strategy.

		Partner	
		Metric	Standard
You	Metric	10, 10	0, 0
	Standard	0, 0	10, 10

Two Nash equilibria.

Unbalanced coordination games

Same scenario as before, except that now you both prefer metric.

		Partner: Metric	Partner: Standard
You	Metric	$10, 10$	$0, 0$
	Standard	$0, 0$	$5, 5$

Still two Nash equilibria.But it is now more likely that both players will choose metric.

Coordination games - Battle of the sexes

Same scenario as before, except that now you prefer metric and your partner prefers standard.

		Partner: Metric	Partner: Standard
You	Metric	$10, 5$	$0, 0$
You	Standard	$0, 0$	$5, 10$

Still two Nash equilibria.Difficult to predict which equilibria will be chosen.

Coordination games - Stag hunt

Same scenario as before, except now you can get partial credit only if you use the standard system.

		Partner: Metric	Partner: Standard
You	Metric	$10, 10$	$0, 5$
	Standard	$5, 0$	$5, 5$

Still two Nash equilibria.If the two don't corrdinate, the player trying for the higher payoff gets penalized.

Anticoordination games - more on multiple equilibria

- Two animals, one piece of food.
- If both animals are passive, they split the food evenly.
- If one animal is aggressive and the other passive, the aggressive one gets most of the food.
- If both animals are aggressive, they destroy the food.

Anticoordination games - more on multiple equilibria

		Animal 2: Aggressive	Animal 2: Passive
Animal 1	Aggressive	$0, 0$	$3, 1$
	Passive	$1, 3$	$2, 2$

Again, two Nash equilibria: (A,P) and (P,A). Just like the game "chicken".

Mixed strategies - matching pennies

- In this game there are two choices H and T.
- Player 1 gets Player 2's penny if they don't match.
- Player 2 gets Player 1's penny if they do match.

		Player 2: H	Player 2: T
Player 1	H	$-1, +1$	$+1, -1$
	T	$+1, -1$	$-1, +1$

There is no Nash equilibrium for this game.

Mixed strategies - matching pennies

Player 2

		H	T
Player 1	H	$-1, +1$	$+1, -1$
	T	$+1, -1$	$-1, +1$

There is no Nash equilibrium for this game.
How would you play in real life?
RANDOMIZE.

- Player 1 chooses to play H with probability p and T with probability $1 - p$.
- Player 2 chooses to play H with probability q and T with probability $1 - q$.
- There are no longer only two strategies for each player. Now there is a set of strategies (*mixed strategies*) corresponding to the interval of numbers from 0 to 1.

Payoffs from mixed strategies

Consider the game from Player 1's perspective:

- Suppose she will choose one of the two *pure strategies* H or T.
- Suppose Player 2 chooses strategy q (play H with probability q and T with probability $1-q$).
- If Player 1 chooses H, then she has a payoff of -1 with probability q and +1 with probability $1-q$. The expected payoff is:

$$(-1) * (q) + (1) * (1-q) = 1 - 2q$$

- If Player 1 chooses T, then she has a payoff of +1 with probability q and -1 with probability $1-q$. The expected payoff is:

$$(1) * (q) + (-1) * (1-q) = 2q - 1$$

Equilibrium with mixed strategies

Is there a Nash equilibrium for the richer version of this game?

Definition

A Nash equilibrium is a pair of strategies (probabilities) such that each is the best response to the other.

What is Player 1's best response to strategy q for Player 2?
If $1 - 2q \neq 2q - 1$, then Player 1 can use a pure strategy H or T, and it will be her best response.
Then Player 2 should change her strategy.
It must be that $q = 1/2$ and $p = 1/2$. This is the Nash equilibrium for this game.

Pareto optimality

What is best for everyone?

Revisit the ice cream example:

		Your brother	
		Ice cream	Mow lawn
You	Ice cream	7, 7	9, 4
	Mow lawn	4, 9	8, 8

Definition

A choice of strategies - one for each player - is *Pareto-optimal* if there is no other choice of strategies in which all players receive payoffs at least as high, and at least one player receives a strictly higher payoff.

Although both getting ice cream is the Nash equilibrium, it is not Pareto optimal.
All other sets of strategies are Pareto optimal.

Social optimality

A stronger condition than Pareto optimality.

Definition

A choice of strategies - one by each player - is *socially optimal* if it maximizes the sum of the players' payoffs.

		Your brother: Ice cream	Your brother: Mow lawn
You	Ice cream	$7, 7$	$9, 4$
	Mow lawn	$4, 9$	$8, 8$

Only (Mow lawn, Mow lawn) is socially optimal.
Note that any socially optimal strategy is also Pareto optimal.