Complexity

CPE 349

Theresa Migler-VonDollen

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- This is one of the most important unsolved problems in mathematics.

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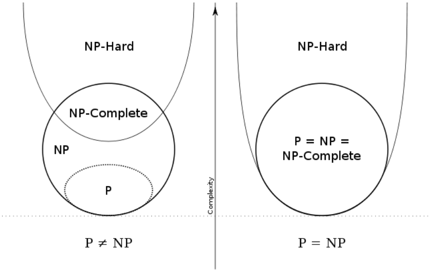

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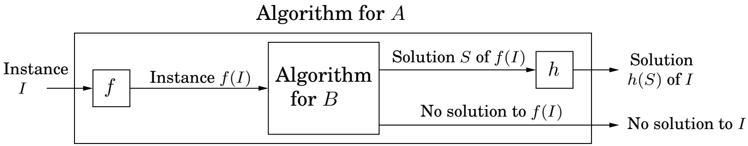

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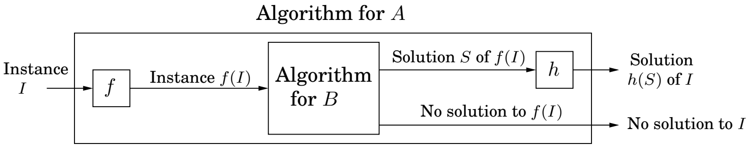

- If such a reduction exists, it implies that B is at least as hard as A.

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$(x \vee y \vee z) \wedge (x \vee \bar{y}) \wedge (y \vee \bar{z}) \wedge (z \vee \bar{x}) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$

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Input: A Boolean formula.
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3-SAT is a special case of k-SAT, when each clause contains exactly $k = 3$ literals. It was one of Karp's 21 NP-complete problems.

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We will use 3-SAT to show that Independent Set is NP-complete.

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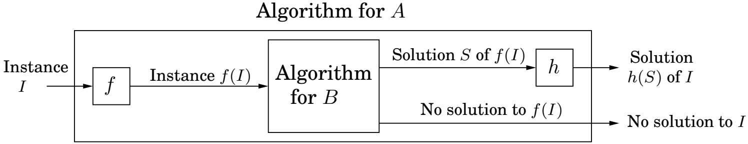
3SAT → Independent Set



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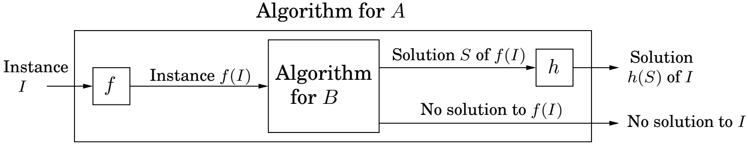

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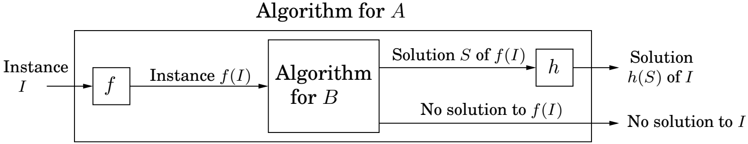

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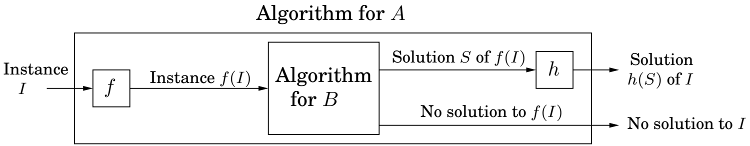

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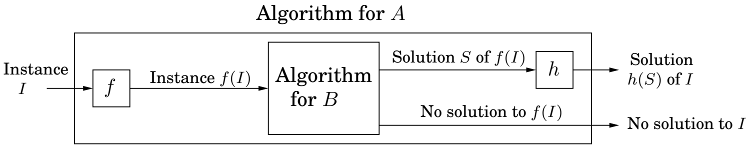

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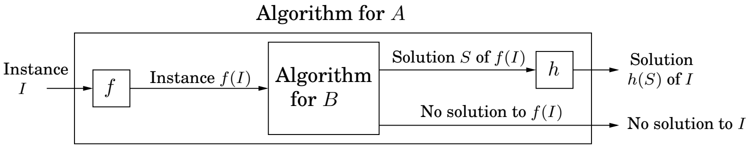

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We need to relate Boolean logic with graphs.

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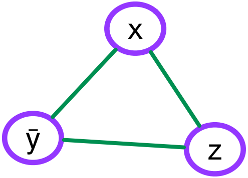

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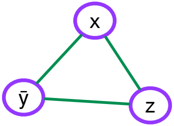

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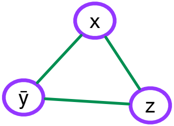

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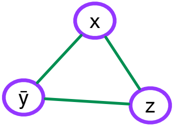

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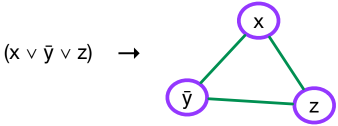

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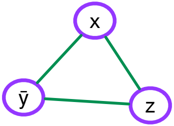

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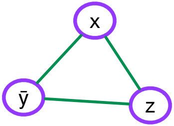

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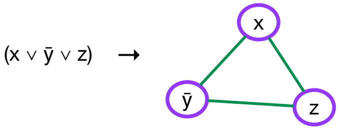

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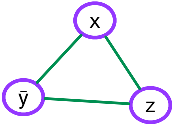
- To force exactly one choice from each clause, take g to be the number of clauses.
  - How can we make sure not to include both x and $\bar{x}$?

3SAT → Independent Set

Example:

$(x \vee \bar{y} \vee z)$ →

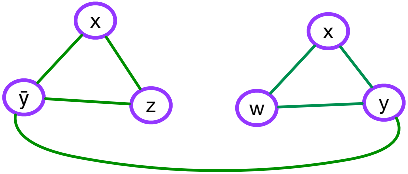


- A triangle has all three vertices maximally connected.
  - This requires that only one vertex is selected for the independent set.
- Repeat this construction for all clauses.
  - The independent set for this graph has to pick at most one literal from each clause.
- To force exactly one choice from each clause, take g to be the number of clauses.
  - How can we make sure not to include both x and $\bar{x}$?
    - Add an edge between every variable and its negation.

3SAT $\rightarrow$ Independent Set

3SAT → Independent Set

$(x \vee \bar{y} \vee z) \wedge (w \vee x \vee y)$ →



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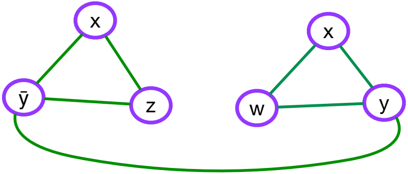

f is the function that takes an input, I, to 3SAT (a boolean formula) and converts it to an input, $f(I)$, for Independent Set (a graph and a value, g).

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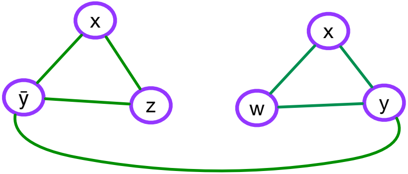

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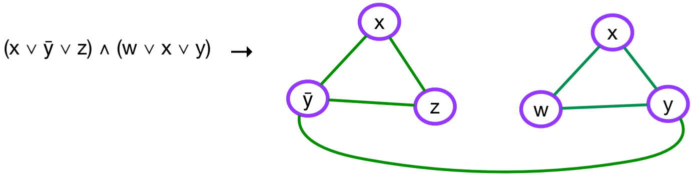
3SAT $\rightarrow$ Independent Set



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3SAT → Independent Set

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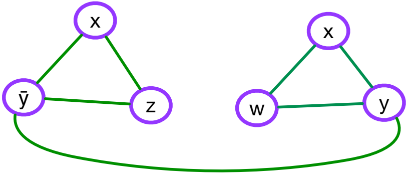

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- For each clause, f creates a triangle where each vertex has label corresponding to each variable in the clause.
- f also adds an edge between vertices representing variable x, and all vertices representing the negation of x, $\bar{x}$.
- If there are n clauses in I, $f(I)$ is the graph described above with value $g = n$.

3SAT $\rightarrow$ Independent Set

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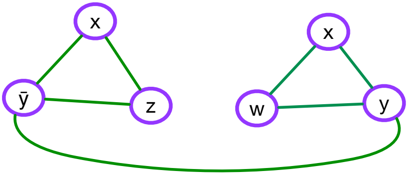
$(x \vee \bar{y} \vee z) \wedge (w \vee x \vee y)$ →



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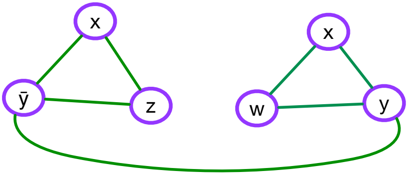

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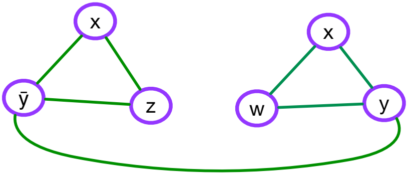

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3SAT $\rightarrow$ Independent Set

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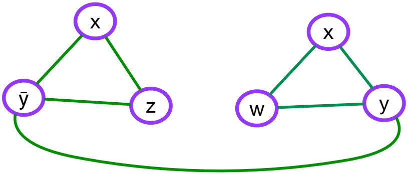

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- A solution for Independent Set is a set of vertices (if this set has size g).
- h simply takes the labels of all of the vertices in the independent set and sets their value to `true`.

3SAT $\rightarrow$ Independent Set

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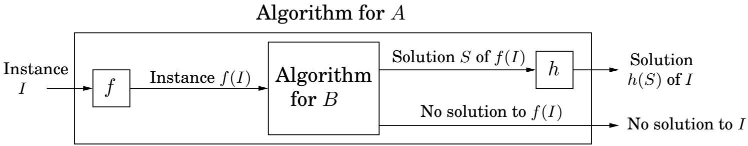

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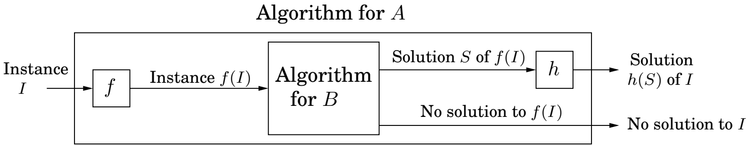
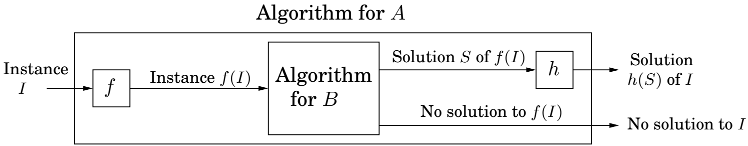


- f is the construction that takes a Boolean formula to a graph and threshold value.

3SAT → Independent Set



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  - This is a polynomial time construction.

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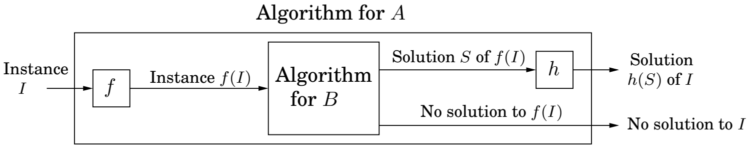

- f is the construction that takes a Boolean formula to a graph and threshold value.
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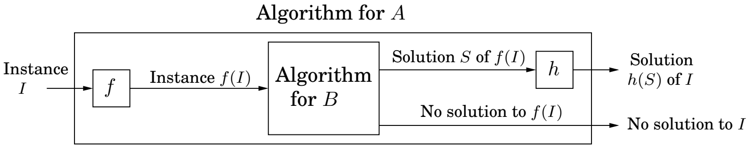
- Suppose that there was a polynomial time algorithm for Independent Set.

3SAT $\rightarrow$ Independent Set



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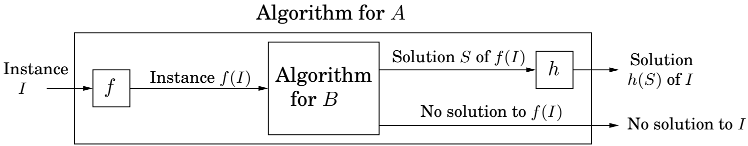

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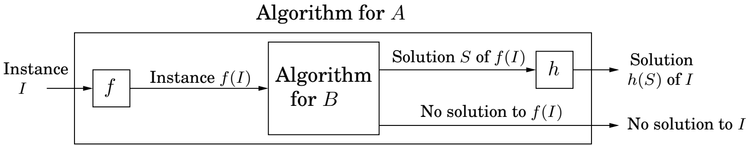

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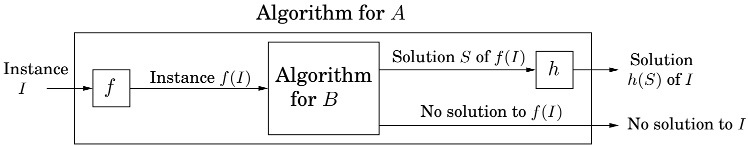
- If G has no independent set of size g, what does that mean?

3SAT $\rightarrow$ Independent Set



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- If G has no independent set of size g, what does that mean?
  - The Boolean formula is not satisfiable.

3SAT $\rightarrow$ Independent Set - Example

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What is the corresponding graph and threshold for the independent set problem?

Vertex Cover

Vertex Cover

Definition

A *vertex cover* is a subset of vertices of a graph such that every edge is incident to at least one vertex in the set.

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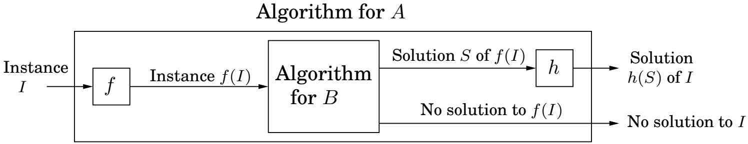
Independent Set $\rightarrow$ Vertex Cover

Independent Set → Vertex Cover



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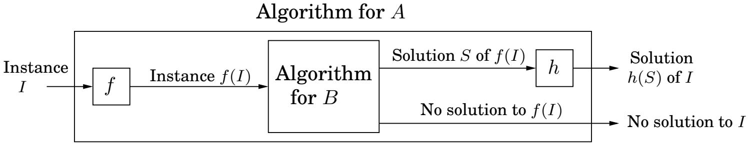
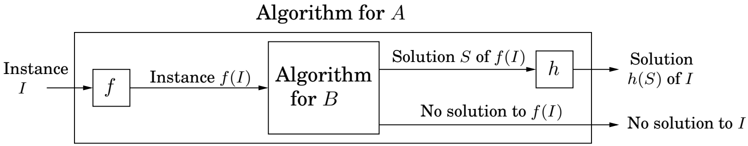


Theorem

A set of vertices S is a vertex cover of a graph G if and only if the vertices $V \setminus S$ are an independent set of G.

Independent Set → Vertex Cover



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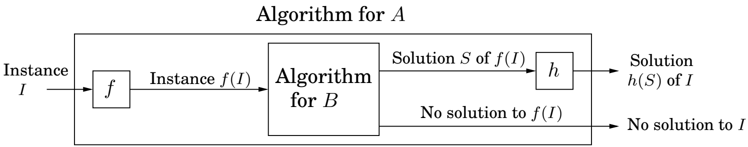

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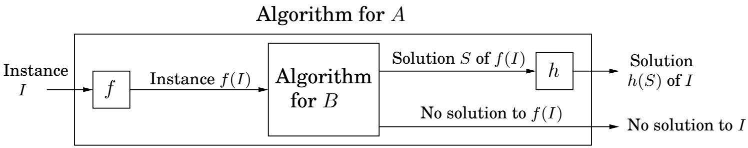

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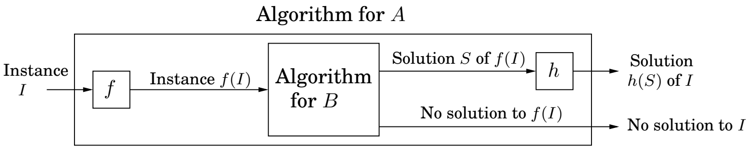

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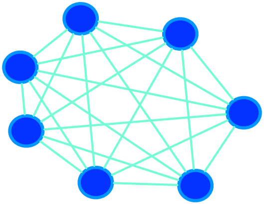

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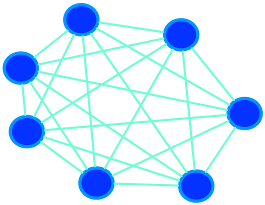

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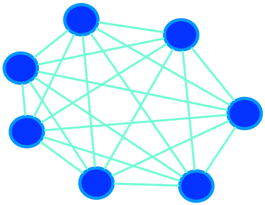

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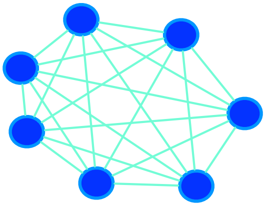

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Input: A graph $G = (V, E)$ and a value k.
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Independent Set $\rightarrow$ Clique

Independent Set $\rightarrow$ Clique

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The *complement* of a graph $G = (V, E)$ is the graph $\bar{G} = (V, \bar{E})$ where $\bar{E}$ contains the possible edges of G that are not in E.

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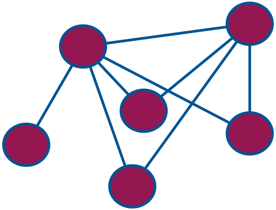

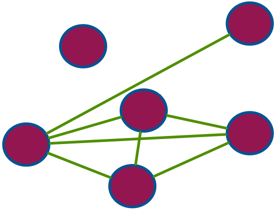

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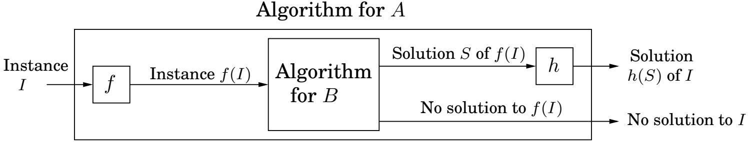

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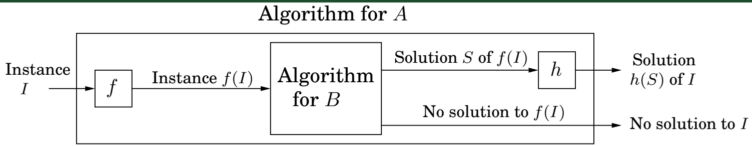

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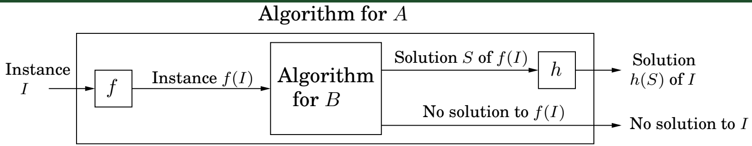

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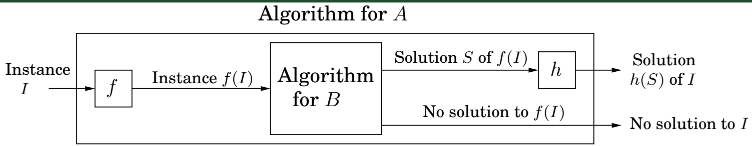

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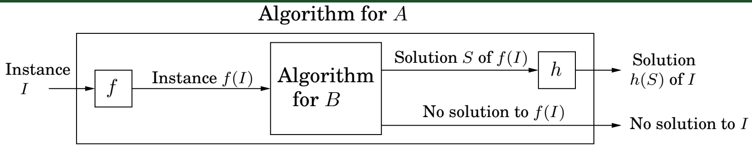

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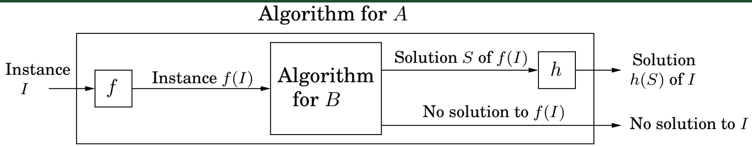

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