Graphs

CPE 103

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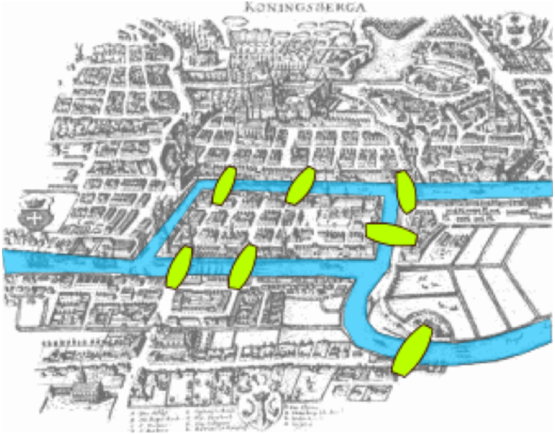
Graph Theory

- Father of graph theory: Leonhard Euler

- Swiss mathematician
- *Seven Bridges of Königsberg* 1736.

Seven Bridges of Königsberg



Is there a walk that traverses each bridge exactly once?

What is a graph?

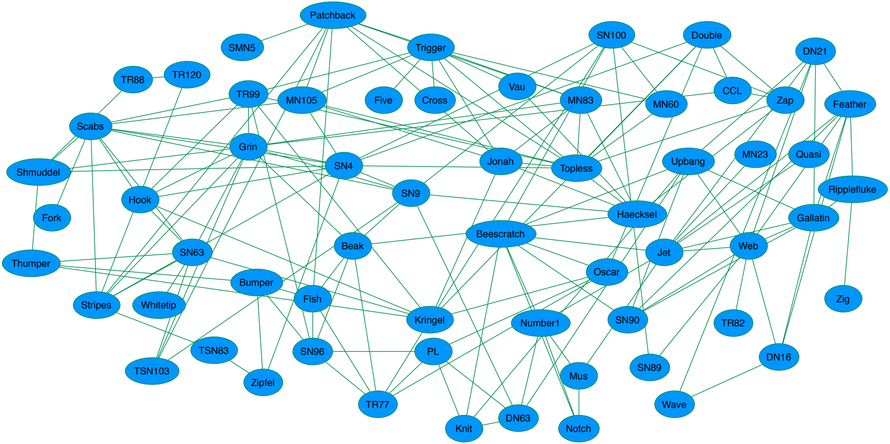
- Vertices and edges.
- Nodes and links.
- People and relationships.

Seven Bridges of Königsberg

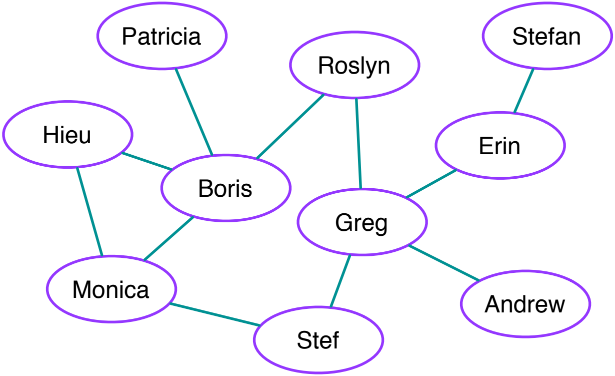
Theorem

There is a walk through a graph that traverses each edge exactly once if and only if the graph is connected and there are exactly two or zero vertices of odd degree.

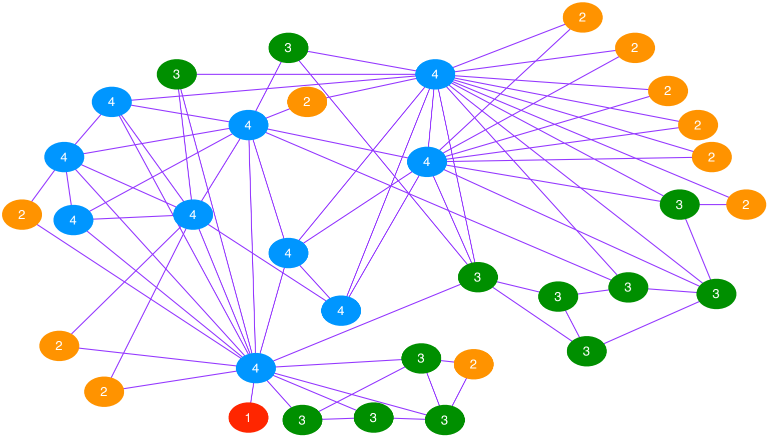
What is a graph? - Dolphin network



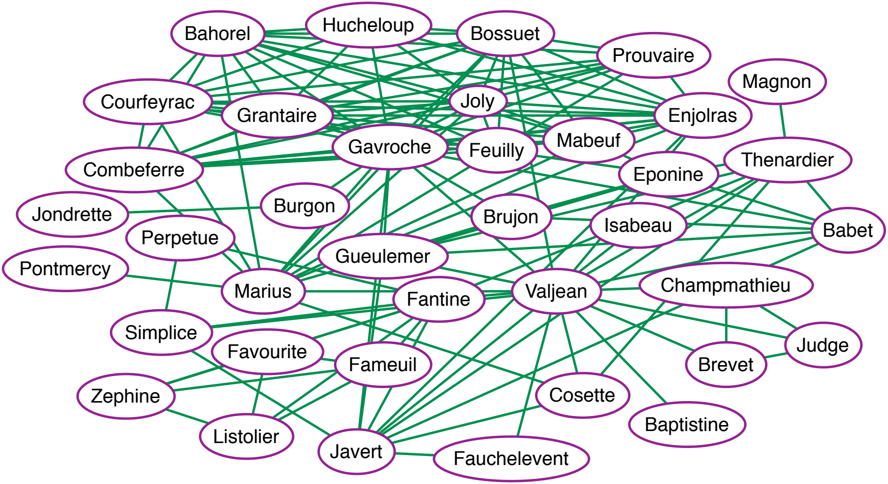
What is a graph? - Friends network



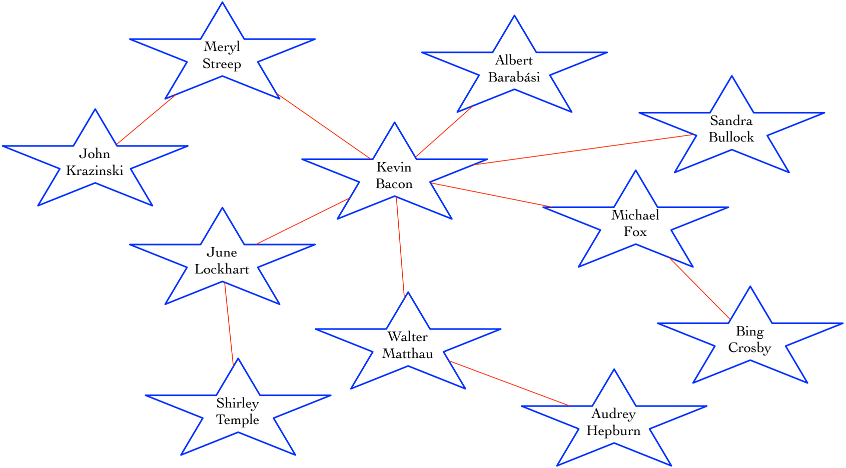
What is a graph? - Karate network



What is a graph? - Les Misérables network



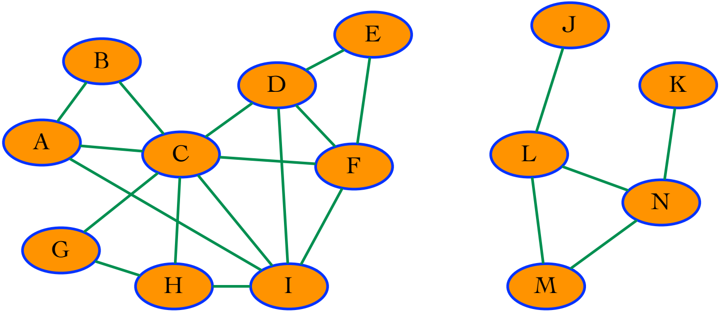
What is a graph? - Actors network



Types of networks

- Collaboration networks
- Who-talks-to-whom graphs
- Information linkage graphs
- Technological networks
- Biological networks

Graph Basics



Definition

A vertex A and a vertex B are *neighbors* if there is an edge, AB, between A and B.

D is neighbors with E, F, and C, but not B.

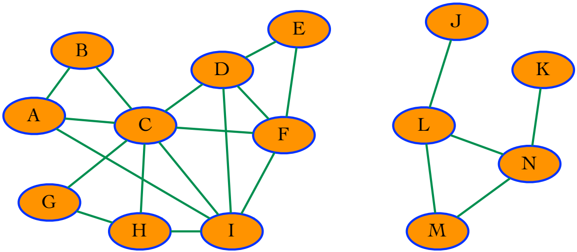
Basic Graph Representations

There are two basic ways to represent a graph, $G = (V, E)$, $V = \{v_1, v_2, \ldots v_n\}$:

1. An *adjacency matrix* is an $n \times n$ array where the (i, j) entry is:
   - $a_{ij} = 1$ if there is an edge from v_i to v_j.
   - $a_{ij} = 0$ otherwise.
2. An *adjacency list* is a set of n linked lists, one for each vertex.
   - The linked list for vertex v holds the names of all vertices, u, such that there is an edge from v to u.

- What is the size of each data structure?
- How long does it take to find a particular edge for each data structure?

Basic ways to describe a graph

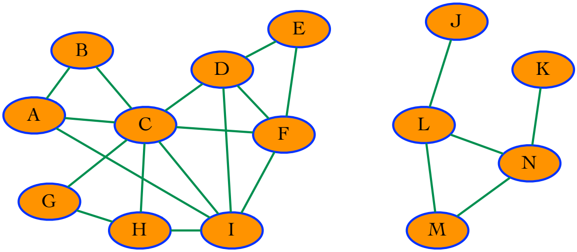


Definition

The *degree* of a vertex is the number of edges adjacent to it (or the number of neighbors).

C has degree 7. J has degree 1.

Graph Basics

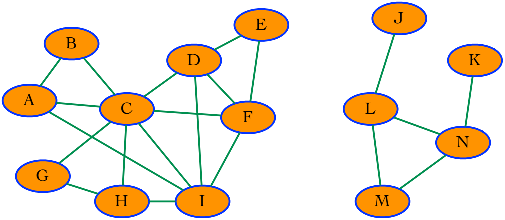


Definition

The *degree distribution* of a graph is the number of vertices of each degree.

$\{0, 2, 4, 4, 2, 1, 0, 1\}$ or $\{0, 1/7, 2/7, 2/7, 1/7, 1/14, 0, 1/14\}$

Graph Basics

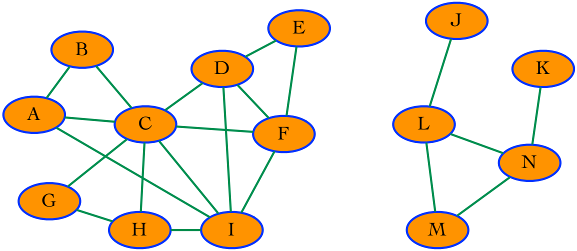


Definition

A *path* between two vertices is a sequence of vertices with the property that each consecutive pair in the sequence is connected by an edge.

There are many paths connecting A and E.
One of these is A, C, D, E, another is A, B, C, G, H, I, F, E.
A, D, E is not a path connecting A and E.

Graph Basics

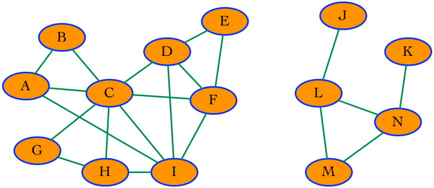


Definition

We say that a graph is *connected* if for each pair of vertices, there is a path between them.

The above graph is not connected.

Graph Basics



Definition

A *connected component* (or just *component*) of a graph is a subset of vertices such that every vertex in the subset has a path to every other vertex in the subset and the subset is not a part of some larger subset with the property that there is a path between every pair of vertices.

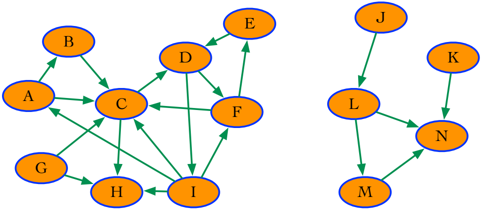
There are two components in the graph $A, B, C, D, E, F, G, H, I$ and J, K, L, M, N. Note that L, M, N is not a component.

Directed Graphs

What if we want to imply one directional relationships?

- Family trees
- Sewage networks
- Food webs
- Webpage network
- Epidemiological networks...

Graph Basics



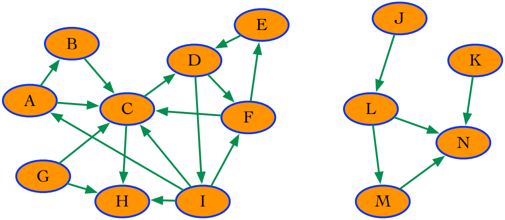
Here $(F, C) \in E$ but $(C, F) \notin E$.

Definition

The *indegree* of a vertex, v, in a directed graph is the number of edges directed into v.
The *outdegree* of a vertex, v, in a directed graph is the number of edges directed out of v.

The indegree of I is 1. The outdegree of I is 4.

Graph Basics

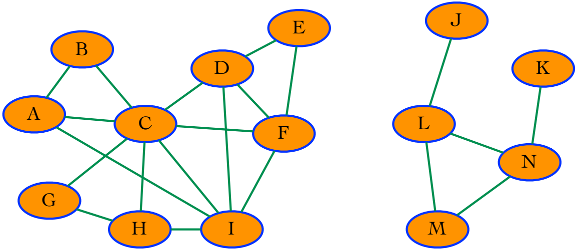


Definition

A *path* in a directed graph from a vertex x to a vertex y is a sequence of vertices with the property that each consecutive pair in the sequence is connected with an edge and all edges are directed in the same direction (out of x).

There is a path from G to E (G, C, D, F, E). There is not a path from H to D.

Graph Basics

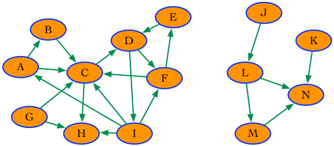


Definition

A *cycle* (in an undirected graph) is a path with at least 3 edges in which the first and last vertices are the same, but otherwise all vertices are distinct.

L, M, N is a cycle, so is A, C, F, I, and many more...

Graph Basics



Definition

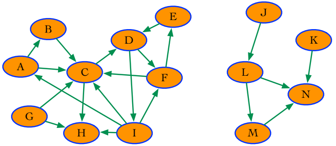
A *cycle* in a directed graph is a (directed) path with at least 2 edges in which the first and last vertices are the same, but otherwise all vertices are distinct.

A, B, C, D, I is a cycle. C, D, E, F, I is not a cycle.

Theorem

A directed graph has a cycle if and only if its DFS tree has a back edge.

Graph Basics

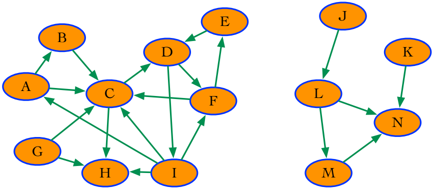


Definition

Two vertices, x and y, are *connected* in a directed graph if there is a path from x to y and y to x.

A and D are connected. L and M are not.

Graph Basics



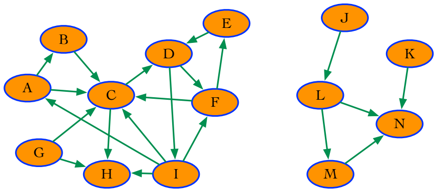
Definition

A directed graph, $G = (V, E)$ is *strongly connected* if for all pairs of vertices $u, v \in V$, u and v are connected.

Definition

The *strongly connected components* of a directed graph partition the graph into strongly connected subgraphs.

Graph Basics



Definition

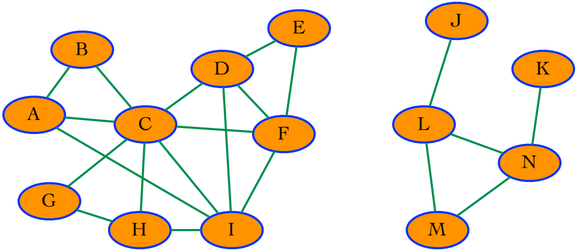
A *directed acyclic graph* or *DAG* is a directed graph with no cycles.

Theorem

Every directed graph is a DAG of its strongly connected components.

We can find such a decomposition in linear time...

Graph Basics



Definition

The *distance* between two vertices is the length of the shortest path connecting them.

The distance between A and F is 2.
By convention, the distance between H and K is ∞.